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Abstract We study the existence and the properties of the reduced measures for the parabolic equations $\partial_t u - \Delta u + g(u) = 0$ in $\Omega \times (0, \infty)$ subject to the conditions (P): $u = 0$ on $\partial\Omega \times (0, \infty)$, $u(x, 0) = \mu$ and (P'): $u = \mu'$ on $\partial\Omega \times (0, \infty)$, $u(x, 0) = 0$ where μ and μ' are positive Radon measures and g a continuous nondecreasing function.

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 1$ and g a nondecreasing continuous function defined on $\mathbb{R}$ and vanishing on $(-\infty, 0]$. This article is concerned with the following question: *Given a positive Radon measure ν on Ω, does it exist a largest Radon measure μ below it for which the initial value problem*

$$\begin{cases} \partial_t u - \Delta u + g(u) = 0 & \text{in } Q_T := \Omega \times (0, T) \\ u = 0 & \text{in } \partial_\ell Q_T := \partial\Omega \times (0, T) \\ u(., 0) = \mu & \text{in } \Omega. \end{cases} \tag{1.1}$$

admits a solution? Whenever μ exists, it is called the *reduced measure* associated to ν. A positive Radon measure for which (1.1) is solvable is called a *good measure*. This type of problems is now well understood for nonlinear elliptic equations. This relaxation phenomenon appeared in the measure framework in the paper [11] by Vazquez dealing with solving the problem

$$-\Delta u + e^{au} = \mu \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

He proved that the reduced measures is the sum of the non-atomic part of μ and the atomic part where the coefficients of the Dirac masses at any atom a are truncated from above at the value $2\pi/a$. Recently the general relaxation problems for the nonlinear elliptic equations

$$\left\{\begin{array}{r l}-\Delta u+g(u)=\mu & \text{in } \Omega\subset\mathbb{R}^N\\ u=0 & \text{in } \partial\Omega\end{array}\right. \tag{1.3}$$

and

$$\left\{\begin{array}{r l}-\Delta u+g(u)=0 & \text{in } \Omega\subset\mathbb{R}^N\\ u=\mu & \text{in } \partial\Omega\end{array}\right. \tag{1.4}$$

are studied respectively by Brezis, Marcus and Ponce [3] and Brezis and Ponce [4]. They prove the existence of a reduced measure μ^* and study its properties, in particular its continuity properties with respect to the capacity $W^{1,2}$ for problem (1.3), or the (N-1)-dimensional Hausdorff measure for problem (1.4).

In this article we study the initial value problem in this perspective and we prove that for any positive bounded Radon measure μ in Ω there exists a largest measure μ^*, smaller than μ such that (1.1) is solvable. We study the set of good measures relative to g and prove that any good measure is absolutely continuous with respect to the Hausdorff measure H^N. In a similar way we study the Cauchy-Dirichlet problem

$$\left\{\begin{array}{r l}\partial_t u-\Delta u+g(u)=0 & \text{in } Q_T:=\Omega\times(0,T)\\ u=\mu & \text{in } \partial_\ell Q_T:=\partial\Omega\times(0,T)\\ u(.,0)=0 & \text{in } \Omega,\end{array}\right. \tag{1.5}$$

and we prove that the reduced measure is absolutely continuous with respect to the same Hausdorff measure H^N.

The proof of many results here follows the ideas borrowed from the theory of reduced measures for elliptic equations as it is developed in [3] and [4]. We choose to expose them for the sake of completeness.

2 Initial value problem

In this section Ω is a bounded domain in $\mathbb{R}^N$ and $\rho(x)=\operatorname{dist}(x,\partial\Omega)$. We denote by $\mathfrak{M}(\Omega)$ the set of Radon measures in Ω and, for $\alpha\in\mathbb{R}$, by $\mathfrak{M}^\alpha(\Omega)$ the subset of $\mu\in\mathfrak{M}(\Omega)$ satisfying

$$\int_\Omega \rho^\alpha(x)\,d\,|\mu|<\infty.$$

Thus $\mathfrak{M}^\alpha_+(\Omega)$ is the positive cone and $\mathfrak{M}^0_+(\Omega)$ the set of bounded measures. For $q\in[1,\infty)$, we denote by $L^q_{\rho^\alpha}(\Omega)$ the corresponding weighted Lebesgue spaces. For $0\leq\tau<\sigma\leq T$ we set $Q_{\tau,\sigma}:=\Omega\times(\tau,\sigma)$, $Q_\sigma:=\Omega\times(0,\sigma)$ and denote by $\partial_\ell Q_{\tau,\sigma}:=\partial\Omega\times(\tau,\sigma]$ and $\partial_\ell Q_\sigma:=\partial\Omega\times(0,\sigma]$ the lateral boundary of these sets. Throughout this paper we make the following assumption on g

$$g \text{ is a nondecreasing continuous function defined on } \mathbb{R} \text{ and vanishing on } (-\infty,0]. \tag{2.1}$$

Definition 2.1 *Let $\mu \in \mathfrak{M}^1_+(\Omega)$. A function $u \in L^1(Q_T)$ is a weak solution of (1.1) in Q_T if $g(u) \in L^1_\rho(Q_T)$ and*

$$\iint_{Q_T} (-u\partial_t\zeta - u\Delta\zeta + \zeta\, g(u))\, dx = \int_\Omega \zeta\, d\mu, \tag{2.2}$$

for all $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$, which is the space of functions in $C^{2,1}(\overline{Q}_T)$ which vanish on $\partial\Omega \times [0,T] \cup \overline{\Omega} \times \{T\}$.

We define in a similar way a weak subsolution (resp. supersolution) of (1.1) by imposing the same integrability conditions on u and $g(u)$ and

$$\iint_{Q_T} (-u\partial_t\zeta - u\Delta\zeta + \zeta\, g(u))\, dx\, dt \leq \int_\Omega \zeta\, d\mu, \tag{2.3}$$

resp.

$$\iint_{Q_T} (-u\partial_t\zeta - u\Delta\zeta + \zeta\, g(u))\, dx\, dt \geq \int_\Omega \zeta\, d\mu, \tag{2.4}$$

for all positive test functions in the same space. More generally we define a subsolution (resp. supersolution) of equation

$$\partial_t u - \Delta u + g(u) = 0 \quad \text{in } Q_T \tag{2.5}$$

as a function $u \in L^1_{loc}(Q_T)$ such that $g(u) \in L^1_{loc}(Q_T)$ and

$$\iint_{Q_T} (-u\partial_t\zeta - u\Delta\zeta + \zeta\, g(u))\, dx\, dt \leq 0, \tag{2.6}$$

resp.

$$\iint_{Q_T} (-u\partial_t\zeta - u\Delta\zeta + \zeta\, g(u))\, dx\, dt \geq 0, \tag{2.7}$$

for all positive test functions ζ in the space $C^{2,1}_0(Q_T)$.

If a solution of (1.1) exists, it is unique, and we shall denote it by u_μ. It is not true that problem (1.1) can be solved for any positive bounded measure μ although it is the case if μ is absolutely continuous with respect to the N-dimensional Hausdorff measure H^N.

Definition 2.2 *A measure for which the problem can be solved is called a good measure relative to g. The subset of $\mathfrak{M}^1_+(\Omega)$ of good measures relative to g is denoted by $\mathcal{G}^\Omega(g)$. If $\mu \in \mathfrak{M}^1_+(\Omega)$ belongs to $\mathcal{G}^\Omega(g)$ for any g satisfying (2.1), is called a universally good measure.*

There are many sufficient conditions which insure the solvability of (1.1), for example

$$\iint_{Q_T} g(\mathbb{E}[\mu])\rho(x) dx\, dt < \infty, \tag{2.8}$$

where $\mathbb{E}[\mu]$ is the heat potential of μ in Ω, that is the solution v of

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } Q_T \\ v = 0 & \text{in } \partial_\ell Q_T \\ v(.,0) = \mu & \text{in } \Omega. \end{cases} \tag{2.9}$$

We recall the parabolic Kato inequality

Lemma 2.3 *Let W be a domain in $\Omega \times \mathbb{R}$, $v \in L^1_{loc}(W)$ and $h \in L^1_{loc}(W)$ such that*

$$-\partial_t v + \Delta v \geq h \quad \textit{in } \mathcal{D}'(W). \tag{2.10}$$

Then

$$-\partial_t v_+ + \Delta v_+ \geq h\chi_{[v\geq 0]} \quad \textit{in } \mathcal{D}'(W). \tag{2.11}$$

Proof. Let $\{\sigma_j\}$ be a regularizing sequence with compact support in the $N+1$ ball $\tilde{B}_{\epsilon_j}(0)$ ($\epsilon_j \to 0$ as $j \to \infty$), and $v_j = v * \sigma_j$. If $V \subset W$ is such that $\operatorname{dist}(V, W^c) > 0$, v_j is defined in V whenever $\epsilon_j < \operatorname{dist}(V, W^c)$. Then

$$-\partial_t v_j + \Delta v_j \geq h_j = h * \sigma_j \quad \text{in } \mathcal{D}'(V), \tag{2.12}$$

and everywhere in V. For $\delta > 0$ let

$$j_\delta(r) = \begin{cases} 0 & \text{if } r < -\delta \\ \dfrac{(r+\delta)^2}{2\delta} & \text{if } -\delta \leq r \leq 0 \\ r + \dfrac{\delta}{2} & \text{if } r > 0. \end{cases}$$

Since

$$-\partial_t j_\delta(v_j) + \Delta j_\delta(v_j) = j'_\delta(v_j)\left(-\partial_t v_j + \Delta v_j\right) + j''_\delta(v_j)|\nabla v_j|^2 \geq j'_\delta(v_j)h_j,$$

and $\phi \in C_0^\infty(W)$ is nonnegative and has compact support in V, it follows that

$$\int_W j_\delta(v_j)\left(\partial_t \phi + \Delta \phi\right) dx\, dt \geq \int_W j'_\delta(v_j) h_j \phi \, dx\, dt.$$

Letting $j \to \infty$, and using the fact that j_δ and j'_δ are continuous and, for some subsequence still denoted $\{\epsilon_j\}$, $\{(v_{\epsilon_j}, h_{\epsilon_j})\}$ converges to (v, h) in L^1_{loc} and almost everywhere in W, we derive from the Lebesgue theorem

$$\int_W j_\delta(v)\left(\partial_t \phi + \Delta \phi\right) dx\, dt \geq \int_W j'_\delta(v) h \phi \, dx\, dt.$$

Now $j_\delta(v)$ converges to v^+ in L^1_{loc} and $j'_\delta(v(x,t))$ converges to 0 if $v(x,t) < 0$ and to 1 if $v(x,t) \geq 0$, i.e. to $\chi_{[v\geq 0]}$. Using again the Lebesgue theorem, we obtain

$$\int_W v^+\left(\partial_t \phi + \Delta \phi\right) dx\, dt \geq \int_W \chi_{[v\geq 0]} h \phi \, dx\, dt,$$

which is (2.11). □

Remark. In an equivalent way, we can state Lemma 2.3 as follows: *If $v \in L^1_{loc}(W)$ and $h \in L^1_{loc}(W)$ are such that*

$$\partial_t v - \Delta v \leq h \quad \textit{in } \mathcal{D}'(W). \tag{2.13}$$

Then

$$\partial_t v_+ - \Delta v_+ \leq h\chi_{[v\geq 0]} \quad \textit{in } \mathcal{D}'(W). \tag{2.14}$$

Definition 2.4 *Let $u \in L^1_{loc}(Q_T)$. 1- We say that u admits the Radon measure μ as an initial trace if it exists*

$$\mathrm{ess\,lim_{t\to 0}}\int_\Omega \mathrm{u(.,t)}\phi\,\mathrm{dx} = \int_\Omega \phi\,\mathrm{d}\mu \quad \forall\phi\in \mathrm{C_0}(\Omega). \tag{2.15}$$

We shall denote $\mu = Tr_\Omega(u)$.
2- We say that u admits the outer regular positive Borel measure $\nu \approx (\mathcal{S},\mu)$ as an initial trace if it exists an open subset $\mathcal{R}\subset\Omega$ and $\mu\in\mathfrak{M}_+(\mathcal{R})$ such that

$$\mathrm{ess\,lim_{t\to 0}}\int_\Omega \mathrm{u(.,t)}\phi\,\mathrm{dx} = \int_\Omega \phi\,\mathrm{d}\mu \quad \forall\phi\in \mathrm{C_0}(\mathcal{R}). \tag{2.16}$$

and, with $\mathcal{S}=\Omega\setminus\mathcal{R}$,

$$\mathrm{ess\,lim_{t\to 0}}\int_\Omega \mathrm{u(.,t)}\phi\,\mathrm{dx} = \infty \quad \forall\phi\in \mathrm{C_0}(\Omega), \phi\geq 0, \phi>0 \textit{ somewhere on } \mathcal{S}. \tag{2.17}$$

We shall denote $\nu = tr_\Omega(u)$.

The trace operator is order preserving. The proof of the following result is straightforward.

Proposition 2.5 *Let u and $\tilde{u}$ in $L^1_{loc}(Q_T)$.*
1- Suppose $Tr_\Omega(u)=\mu$ and $Tr_\Omega(\tilde{u})=\tilde{\mu}$. Then

$$\tilde{u}\leq u \Longrightarrow \tilde{\mu}\leq\mu. \tag{2.18}$$

2- Suppose $tr_\Omega(u)=\nu\approx(\mathcal{S},\mu)$ and $tr_\Omega(\tilde{u})=\tilde{\nu}\approx(\tilde{\mathcal{S}},\tilde{\mu})$. Then

$$\tilde{u}\leq u \Longrightarrow \tilde{\mathcal{S}}\subset\mathcal{S} \quad \textit{and } \tilde{\mu}\lfloor_{\mathcal{S}^c}\leq\mu\lfloor_{\mathcal{S}^c}. \tag{2.19}$$

The next classical results characterize the nonnegative supersolutions or subsolutions. We give their proof for the sake of completeness.

Proposition 2.6 *Let $u\in L^1(Q_T)$ be a nonnegative supersolution of (2.5) in Q_T such that $g(u)\in L^1(Q_T)$. Then there exists a positive Radon measure μ such that $\mu = Tr_\Omega(u)$.*

Proof. If $0<\sigma<\tau<T$ are two Lebesgue points of $t\mapsto \|u(.,t)\|_{L^1}$ and $\phi\in C^2_0(\Omega)$, $\phi\geq 0$, we set $Q_{\sigma,\tau}=\Omega\times(\sigma,\tau)$, take $\zeta(x,t)=\chi_{[\sigma,\tau]}(t)\phi(x)$ (by approximations) and derive from the definition that

$$\int_\Omega u(.,\tau)\phi\,dx - \int_\Omega u(.,\sigma)\phi\,dx + \iint_{Q_{\sigma,\tau}}(-u\Delta\zeta+\zeta\,g(u))\,dx\,dt\geq 0. \tag{2.20}$$

Set

$$H(\sigma)=\iint_{Q_{\sigma,\tau}}(-u\Delta\phi+\phi\,g(u))\,dx\,dt$$

Then $H\in L^1(0,\tau)$ and the mapping

$$\sigma\mapsto\Psi(\sigma)=\int_\Omega u(.,\sigma)\phi\,dx - H(\sigma)$$

is a.e. nondecreasing on $(0,\tau]$ and it admits an essential limit $L(\phi)\in\mathbb{R}$ as $\sigma\to 0$. Therefore it exists

$$\ell(\phi)=\mathrm{ess}\lim_{\sigma\to 0}\int_\Omega \mathrm{u}(.,\sigma)\phi\,\mathrm{dx},$$

and the mapping $\phi\mapsto\ell(\phi)$ defines a positive Radon measure μ in Ω. $\square$

It is possible to get rid of the integrability assumption on u if it is assumed that u vanishes on the boundary and Ω is bounded.

Proposition 2.7 *Let u be a positive supersolution of (2.5) in Q_T which vanishes on $\partial_\ell Q_T$ in the sense that (2.4) holds for all nonnegative $\zeta\in C^{2,1}_{\ell,0}(\overline{Q}_T)$. If $g(u)\in L^1_\rho(Q_T)$, there exists $\mu\in\mathfrak{M}^1_+(\Omega)$ such that $\mu=Tr_\Omega(u)$.*

Proof. As a test function we take $\zeta(x,t)=\chi_{[\sigma,\tau]}(t)\phi_1(x)$ where ϕ_1 is the first eigenfunction of $-\Delta$ in $W^{1,2}_0(\Omega)$, $\phi_1\geq 0$ and λ_1 the corresponding eigenvalue. Thus (2.20) is replaced by

$$\int_\Omega u(.,\tau)\phi_1\,dx-\int_\Omega u(.,\sigma)\phi_1\,dx+\iint_{Q_{\sigma,\tau}}(\lambda_1 u+g(u))\,\phi_1 dx\,dt\geq 0. \tag{2.21}$$

If we set

$$X(\tau)=\iint_{Q_{\sigma,\tau}}u\phi_1\,dx\,dt,$$

and

$$G(\sigma)=\iint_{Q_{\sigma,\tau}}\phi_1\,g(u)\,dx\,dt,$$

then (2.21) reads as

$$X'(\sigma)+\lambda_1 X(\sigma)+G(\sigma)\geq X'(\tau)\quad\text{a.e. }0<\sigma<\tau,$$

which yields to

$$\frac{d}{d\sigma}\left(e^{\lambda_1\sigma}X(\sigma)-\int_\sigma^\tau e^{\lambda_1 t}\left(G(t)-X'(\tau)\right)dt\right)\geq 0.$$

The conclusion follows as in Proposition 2.6. Notice also that another choice of test function yields to $u\in L^1(\Omega)$. $\square$

For subsolutions of (2.5) we prove the following.

Proposition 2.8 *Let $u\in L^1(Q_T)$ be a nonnegative subsolution of (2.5) in Q_T such that $g(u)\in L^1(Q_T)$. Then there exists a positive outer regular Borel measure ν on Ω such that $\nu=tr_\Omega(u)$.*

Proof. Defining H as in the proof of Proposition 2.6 we obtain that

$$\sigma\mapsto\Psi(\sigma)=\int_\Omega u(.,\sigma)\phi\,dx+H(\sigma)$$

is nonincreasing on $(0,\tau]$ and it admits a limit $L^*(\phi)\in(-\infty,\infty]$ as $\sigma\to 0$. For any $\xi\in\Omega$ the following dichotomy holds,

(i) either there exists a $\phi \in C_0^2(\Omega)$ verifying $\phi(\xi) > 0$ such that $L(\phi) < \infty$,
(ii) or for any $\phi \in C_0^2(\Omega)$ verifying $\phi(\xi) > 0$, $L(\phi) = \infty$.
The set $\mathcal{R}(u)$ of ξ such that (i) occurs is open and there exists $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ such that

$$L(\phi) = \int_{\mathcal{R}(u)} \phi \, d\mu \quad \forall \phi \in C_0(\mathcal{R}(u)).$$

The set $\mathcal{S}(u) = \Omega \setminus \mathcal{S}(u)$ is relatively closed in Ω. Further, if $\phi \in C_0(\Omega)$ is nonnegative and positive somewhere on $\mathcal{S}(u)$, there holds

$$\mathrm{ess}\lim_{\mathrm{t}\to 0} \int_{\Omega} \mathrm{u}(.,\sigma)\phi \, \mathrm{dx} = \infty.$$

The outer regular Borel measure ν is defined for any Borel subset $E \subset \Omega$ by

$$\nu(E) = \begin{cases} \displaystyle\int_E d\mu & \text{if } E \subset \mathcal{R}(u) \\ \infty & \text{if } E \cap \mathcal{S}(u) \neq \emptyset. \end{cases}$$

□

The next lemma is the parabolic counterpart of an elliptic result proved in [4]

Lemma 2.9 *Let $f \in L^1_\rho(Q_T)$ and $u \in L^1(Q_T)$ such that*

$$\iint_{Q_T} u(\partial_t \zeta + \Delta\zeta) dx\, dt = -\iint_{Q_T} f\zeta \, dx\, dt \tag{2.22}$$

for every $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$. Then

$$\lim_{n\to\infty} n \iint_{Q_T \cap \{\rho(x) \leq n^{-1}\}} |u| \, dx\, dt = 0. \tag{2.23}$$

Proof. We assume first that $f \geq 0$, then $u \geq 0$. Let H be a nondecreasing concave C^2 function such that $H(0) = 0$, $H''(t) = -1$ for $0 \leq t \leq 1$ and $H(t) = 1$ for $t \geq 2$. Let ξ_0 be the solution of

$$\begin{cases} \partial_t \xi_0 + \Delta \xi_0 = -1 \quad \text{in } Q_T \\ \qquad \xi_0(.,T) = 0 \quad \text{in } \bar{\Omega} \\ \qquad \xi_0(x,t) = 0 \quad \text{in } \partial\Omega \times [0,T]. \end{cases} \tag{2.24}$$

Let $w_n = n^{-1} H(n\xi_0)$, then

$$-\partial w_n - \Delta w_n \geq -nH''(n\xi_0) \left|\nabla \xi_0\right|^2 \geq n\chi_{\{\xi_0 \leq n^{-1}\}} \left|\nabla \xi_0\right|^2.$$

Therefore

$$\iint_{Q_T} f w_n dx\, dt = -\iint_{Q_T} u(\partial_t w_n + \Delta w_n) dx\, dt \geq n \iint_{Q_T} \left|\nabla \xi_0\right|^2 u dx\, dt.$$

But $w_n \leq \min\{\xi_0, n^{-1}\}$, therefore, by the Lebesgue theorem,

$$0 = \lim_{n\to\infty} \iint_{Q_T} f w_n dx\, dt = \lim_{n\to\infty} n \iint_{Q_T} |\nabla \xi_0|^2 \, u dx\, dt.$$

Let $\epsilon > 0$, by Hopf lemma on $Q_{T-\epsilon}$, there exists $c_1 > 0$, $c_2 > 0$ such that $|\nabla \xi_0| \geq c_1$ on $\partial\Omega \times [0, T-\epsilon]$; thus $c_2 \xi_0 \leq \rho \leq c_2^{-1} \xi_0$ and

$$\lim_{n\to\infty} n \iint_{Q_{T-\epsilon} \cap \{\xi_0(x) \leq n^{-1}\}} u dx\, dt = 0.$$

Clearly we can extend f to be zero for $t > T$ and $\tilde{u}$ to be the weak solution of

$$\left\{ \begin{array}{l} \partial_t \tilde{u} + \Delta \tilde{u} = 0 \quad \text{in } Q_{T,T+\epsilon} \\ \qquad \tilde{u}(.,T) = u(.,T) \quad \text{in } \bar{\Omega} \\ \qquad \tilde{u}(x,t) = 0 \quad \text{in } \partial\Omega \times [T, T+\epsilon]. \end{array} \right.$$

Notice that it is always possible to assume that T is a Lebesque point of $t \mapsto \|u(.,t)\|_{L^1}$ inasmuch this function is actually continuous. Replacing T by $T+\epsilon$, we derive (2.23). Next, if u has not constant sign, we denote by v the weak solution of

$$\left\{ \begin{array}{l} \partial_t v - \Delta v = |f| \quad \text{in } Q_T \\ \qquad v(.,0) = 0 \quad \text{in } \bar{\Omega} \\ \qquad v(x,t) = 0 \quad \text{in } \partial\Omega \times [0,T]. \end{array} \right.$$

Then $|u| \leq v$ and the proof follows from the first case. $\square$

Lemma 2.10 *Let $f \in L^1_\rho(Q_T)$ and $u \in L^1(Q_T)$ such that*

$$-\iint_{Q_T} u(\partial_t \zeta + \Delta \zeta) dx\, dt \leq \iint_{Q_T} f \zeta \, dx\, dt \tag{2.25}$$

for every $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$, $\zeta \geq 0$. Then, for the same class of test functions ζ, there holds

$$-\iint_{Q_T} (\partial_t \zeta + \Delta \zeta) u_+ dx\, dt \leq \iint_{Q_T \cap \{u \geq 0\}} f \zeta \, dx\, dt. \tag{2.26}$$

Proof. By Lemma 2.3, (2.26) holds for any $\zeta \in C^{2,1}_0(Q_T)$. Let $\{\gamma_n\}$ be a sequence of functions in $C^{2,1}_0(Q_T)$ such that $0 \leq \gamma_n \leq 1$, $\gamma_n(x,t) = 1$ if $\rho(x) \geq n^{-1}$ or $t \geq n^{-1}$, $\|\nabla \gamma_n\|_{L^\infty} \leq Cn$, $\|\Delta \gamma_n\|_{L^\infty} \leq Cn^2$ and $\|\partial_t \gamma_n\|_{L^\infty} \leq Cn$. Given $\zeta \geq 0$ in $C^{2,1}_{\ell,0}(\overline{Q}_T)$, $\zeta \gamma_n$ is an admissible test function for Kato's inequality (2.26), thus

$$-\iint_{Q_T} (\partial_t(\zeta\gamma_n) + \Delta(\zeta\gamma_n)) u_+ dx\, dt \leq \iint_{Q_T \cap \{u \geq 0\}} f(\zeta\gamma_n) \, dx\, dt. \tag{2.27}$$

When $n \to \infty$ the right-hand side of (2.27) converges to the right-hand side of (2.26). Moreover $\partial_t(\zeta\gamma_n) = \gamma_n\partial_t\zeta + \zeta\partial_t\gamma_n$, $\nabla(\zeta\gamma_n) = \gamma_n\nabla\zeta + \zeta\nabla\gamma_n$ and $\Delta(\zeta\gamma_n) = \gamma_n\Delta\zeta + \zeta\Delta\gamma_n + 2\nabla\zeta.\nabla\gamma_n$. Thus

$$\partial_t(\zeta\gamma_n) + \Delta(\zeta\gamma_n) = \gamma_n\partial_t\zeta + \zeta\partial_t\gamma_n + \gamma_n\Delta\zeta + \zeta\Delta\gamma_n + 2\nabla\zeta.\nabla\gamma_n.$$

Since ζ vanishes on $\partial\Omega \times [0, T]$ and is bounded with bounded gradient, there holds

$$\left|\iint_{Q_T} (\zeta\partial_t\gamma_n + \zeta\Delta\gamma_n + 2\nabla\zeta.\nabla\gamma_n)\, u^+ \, dx\, dt\right| \leq Cn \iint_{Q_T\cap\{\rho(x)\leq n^{-1}\}} u^+ \, dx\, dt$$

which goes to 0 as $n \to \infty$. This implies (2.26). $\square$

If we deal with subsolution or supersolutions of problem (1.1) we have the following results

Theorem 2.11 *Let $\mu \in \mathfrak{M}^1_+(\Omega)$ and u be a nonnegative subsolution of (1.1). Then the initial trace of u is a positive Radon measure $\tilde{\mu}$ such that $\tilde{\mu} \leq \mu$. Furthermore, if (1.1) admits a weak solution u_μ, there holds $u \leq u_\mu$.*

Proof. Step 1. There holds $\tilde{\mu} \leq \mu$. If σ is a Lebesgue point of $t \mapsto \|\tilde{u}(., t)\|_{L^1}$ and $\phi \in C^2_0(\Omega)$, $\phi \geq 0$, we can take $\zeta(x, t) = \chi_{[0,\sigma]}(t)\phi(x)$ (by approximations) and derive from (2.3) that

$$\int_\Omega u(., \sigma)\phi \, dx + \iint_{Q_\sigma} (-u\Delta\zeta + \zeta\, g(u))\, dx\, dt \leq \int_\Omega \phi \, d\mu, \tag{2.28}$$

thus, by Proposition 2.8, using the fact that $u \in L^1(Q_T)$ and $g(u) \in L^1_\rho(Q_T)$,

$$\text{ess}\lim_{\sigma\to 0} \int_\Omega \mathrm{u}(., \sigma)\phi \, \mathrm{dx} \leq \int_\Omega \phi \, \mathrm{d}\mu. \tag{2.29}$$

It follows that the initial trace $\tilde{\nu} \approx (\mathcal{S}(u), \tilde{\mu}$ has no singular part ($\mathcal{S}(u) = \emptyset$) and $\tilde{\mu} \leq \mu$. This implies that $\phi \mapsto m(\phi)$ is a measure dominated by μ that we shall denote by $\tilde{\mu}$. It represents the initial trace of $\tilde{u}$, and we shall denote it by

$$\tilde{\mu} = Tr_\Omega(\tilde{u}). \tag{2.30}$$

Next we take $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$, $\zeta \geq 0$, and get at any Lebesgue point σ as in Proposition 2.6-Proposition 2.8

$$\iint_{Q_{\sigma,T}} (-u\partial_t\zeta - u\Delta\zeta + \zeta\, g(u))\, dx\, dt \leq \int_\Omega u(., \sigma)\zeta \, dx,$$

we derive, by letting $\sigma \to 0$,

$$\iint_{Q_T} (-u\partial_t\zeta - u\Delta\zeta + \zeta\, g(u))\, dx\, dt \leq \int_\Omega \zeta(., 0)\, d\tilde{\mu}. \tag{2.31}$$

Step 2. There exists $u_{\tilde\mu}$ and $u_{\tilde\mu} \leq u_\mu$. For $k > 0$ set $g_k(r) = \min\{g(r), k\}$ and let $u = u^k_{\tilde\mu}$ be the solution of

$$\left\{\begin{aligned} \partial_t u - \Delta u + g_k(u) &= 0 \quad \text{in } Q_T := \Omega \times (0,T) \\ u &= 0 \quad \text{in } \partial_\ell Q_T := \partial\Omega \times (0,T) \\ u(.,0) &= \tilde\mu \quad \text{in } \Omega. \end{aligned}\right. \tag{2.32}$$

Defining in the same way u^k_μ, we obtain $u^k_{\tilde\mu} \leq u^k_\mu$, $u^k_{\tilde\mu} \geq u^{k'}_{\tilde\mu}$ and $u^k_\mu \geq u^{k'}_\mu \geq u_\mu$ for $k' > k > 0$. If $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$ is nonnegative, there holds

$$\iint_{Q_T} \zeta\, g_k(u^k_\mu) dx\, dt = \int_\Omega \zeta\, d\mu + \iint_{Q_T} (\partial_t\zeta + \Delta\zeta)\, u^k_\mu dx\, dt. \tag{2.33}$$

Clearly u^k_μ converges to some $U \geq u_\mu$ when $k \to \infty$, the right-hand side of (2.33) converges to

$$\int_\Omega \zeta\, d\mu + \iint_{Q_T} (\partial_t\zeta + \Delta\zeta)\, U dx\, dt,$$

and $g_k(u^k_\mu)$ converges to $g(U)$ a. e. By Fatou

$$\iint_{Q_T} \zeta\, g(U) dx\, dt \leq \liminf_{k\to\infty} \iint_{Q_T} \zeta\, g_k(u^k_\mu) dx\, dt,$$

thus, using the monotonicity of g,

$$\iint_{Q_T} \zeta\, g(u_\mu) dx\, dt \leq \iint_{Q_T} \zeta\, g(U) dx\, dt \leq \int_\Omega \zeta\, d\mu + \iint_{Q_T} (\partial_t\zeta + \Delta\zeta)\, U dx\, dt. \tag{2.34}$$

Because u_μ satisfies (2.2), all the three terms in (2.34) are equal, $U = u_\mu$ and

$$\lim_{k\to\infty} \iint_{Q_T} \zeta\, g_k(u^k_\mu) dx\, dt = \iint_{Q_T} \zeta\, g(u_\mu) dx\, dt. \tag{2.35}$$

Next $u^k_{\tilde\mu}$ decreases and converges to some $\tilde U$, $g_k(u^k_{\tilde\mu}) \to g(\tilde U)$ a.e., and

$$\iint_{Q_T} \zeta\, g(\tilde U) dx\, dt \leq \lim_{k\to\infty} \iint_{Q_T} \zeta\, g_k(u^k_{\tilde\mu}) dx\, dt = \int_\Omega \zeta\, d\tilde\mu + \iint_{Q_T} (\partial_t\zeta + \Delta\zeta)\, \tilde U dx\, dt. \tag{2.36}$$

Since $0 \leq \zeta\, g_k(u^k_{\tilde\mu}) \leq \zeta\, g_k(u^k_\mu)$. In order to prove that

$$\lim_{k\to\infty} \iint_{Q_T} \zeta\, g_k(u^k_{\tilde\mu}) dx\, dt = \iint_{Q_T} \zeta\, g(\tilde U) dx\, dt, \tag{2.37}$$

we use the following classical result : *Let $h_n \geq \tilde h_n \geq 0$ two sequences of measurable functions in some measured space (G, Σ, dm) which converge a. e. in G to h and $\tilde h$ respectively. Then*

$$\lim_{n\to\infty} \int_G h_n dm = \int_G h dm \Longrightarrow \lim_{n\to\infty} \int_G \tilde h_n dm = \int_G \tilde h dm.$$

Therefore (2.35) implies (2.37). From (2.36) we get

$$\iint_{Q_T} \zeta\, g(\tilde U) dx\, dt = \int_\Omega \zeta\, d\tilde\mu + \iint_{Q_T} (\partial_t \zeta + \Delta\zeta)\, \tilde U dx\, dt. \tag{2.38}$$

This relation is valid with any $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$ with constant sign. It implies in particular that $Tr_\Omega(\tilde U) = \tilde\mu$. Thus $u_{\tilde\mu}$ exists and $\tilde U = u_{\tilde\mu}$.
Step 3. We claim that $u \leq u_{\tilde\mu}$. Set $w = u - u_{\tilde\mu}$, it follows from (2.31),

$$\iint_{Q_T} (-w\partial_t\zeta - w\Delta\zeta + (g(u) - g(u_{\tilde\mu}))\zeta)\, dx\, dt \leq 0 \tag{2.39}$$

for any $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$, $\zeta \geq 0$. Using Lemma 2.10 we derive

$$\iint_{Q_T \cap \{w^+ \geq 0\}} (-(\partial_t\zeta - \Delta\zeta) w_+ + +(g(u) - g(u_{\tilde\mu}))\zeta)\, dx\, dt \leq 0 \tag{2.40}$$

We take $\zeta = \xi_0$ given by (2.24). Since g is nondecreasing, we derive

$$\iint_{Q_T \cap \{w^+ \geq 0\}} w_+ dx\, dt \leq 0. \tag{2.41}$$

Thus $u \leq u_{\tilde\mu} \leq u_\mu$. □

Remark. It is noticeable that Step-2 of the proof of Theorem 2.11 can be stated in the following way. *If* $\mu \in \mathfrak{M}^1_+(\Omega)$ *is a good measure, any measure* $\tilde\mu$ *such that* $0 \leq \tilde\mu \leq \mu$ *is a good measure.*

Consider $\mu \in \mathfrak{M}^1_+(\Omega)$. The relaxation phenomenon associated to (1.1) can be constructed in the following way. Let $\{g_k\}$ be an increasing sequence of continuous nondecreasing functions defined on $\mathbb{R}$, vanishing on $(-\infty, 0]$ and such that

$$\begin{array}{ll} (i) & 0 \leq g_k(r) \leq c_k r^p + c'_k \quad \forall r \geq 0, \ \ \forall k > 0 \\ (ii) & \displaystyle\lim_{k\to\infty} g_k(r) = g(r) \quad \forall r \in \mathbb{R}, \end{array} \tag{2.42}$$

for some positive constants c_k and c'_k and $p \in (1, (N+2)/(N+1))$. Since (2.8) is satisfied, there exists a unique solution $u = u_k$ to

$$\left\{ \begin{aligned} \partial_t u - \Delta u + g_k(u) = 0 \quad & \text{in } Q_T \\ u = 0 \quad & \text{in } \partial_\ell Q_T \\ u(.,0) = \mu \quad & \text{in } \Omega. \end{aligned} \right. \tag{2.43}$$

It is noticeable that, if the assumption $\mu \in \mathfrak{M}^1_+(\Omega)$ were replaced by $\mu \in \mathfrak{M}^0_+(\Omega)$, the exponent p in (2.42) should have been taken smaller than $(N+2)/N$. In the sequel C will denote a positive constant, depending on the data, not on k, the value of which may change from one occurrence to another. Our first result points out the relaxation phenomenon associated to the sequence $\{u_k\}$.

Theorem 2.12 *When $k \to \infty$, the sequence $\{u_k\}$ converges in $L^1(Q_T)$ to a some nonnegative function u^* such that $g(u^*) \in L^1_\rho(Q_T)$, and there exists a positive measure μ^* smaller that μ with the property that*

$$\begin{cases} \partial_t u^* - \Delta u^* + g(u^*) = 0 & \text{in } Q_T \\ u^* = 0 & \text{in } \partial_\ell Q_T \\ u^*(.,0) = \mu^* & \text{in } \Omega. \end{cases} \tag{2.44}$$

Furthermore u^* is the largest subsolution of problem (1.1).

Proof. By [7, Lemma1.6] there holds

$$\|u_k\|_{L^1} + \|g_k(u_k)\|_{L^1_\rho} \leq C \int_\Omega \rho \, d\mu, \tag{2.45}$$

and, by the maximum principle,

$$u_k \leq \mathbb{E}[\mu] \quad \text{in } \; Q_T. \tag{2.46}$$

For any $\epsilon > 0$ we denote $Q_{\epsilon,T} = \Omega \times [\epsilon, T]$. Since $\mathbb{E}[\mu]$ is uniformly bounded in $Q_{\epsilon,T}$ for any $\epsilon > 0$, it follows by the parabolic equations regularity theory that, u_k is bounded in $C^{1+\alpha,\alpha/2}(Q_{\epsilon,T})$ for any $0 < \alpha < 1$. Furthermore, if $k' > k$, $g_{k'}(u_k) \geq g_k(u_k)$ thus u_k is a super-solution for the equation satisfied by $u_{k'}$. This implies $u_k \geq u_{k'}$ and $u^* := \lim_{k\to\infty} u_k$ exists and satisfies

$$u^* \leq \mathbb{E}[\mu] \quad \text{in } \; Q_T.$$

Because of (2.46) uniform boundedness holds also in $L^p(Q_T)$, for any $p \in [1, (N+2)/(N+1))$. By the Lebesgue theorem the convergence occurs in $L^p(Q_T)$ too, for any $p \in [1, (N+2)/(N+1))$, and locally uniformly in Q_T by the standard regularity theory. By continuity $g_k(u_k)$ converges to $g(u^*)$ uniformly in $Q_{\epsilon,T}$, thus u^* satisfies

$$\partial_t u^* - \Delta u^* + g(u^*) = 0 \quad \text{in } Q_T$$

and vanishes on $\partial_\ell Q_T$. By the Fatou theorem

$$\iint_{Q_T} g(u^*)\zeta \, dx \, dt \leq \liminf_{k\to\infty} \iint_{Q_T} g_k(u_k)\zeta \, dx \, dt,$$

for any $\zeta \in C(\overline{Q}_T)$, $\zeta \geq 0$, and there exists a positive measure λ in Q_T such that

$$g_k(u_k) \to g(u^*) + \lambda,$$

weakly in the sense of measures. Thus for any $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$, there holds

$$\iint_{Q_T} (-u^*\partial_t\zeta - u^*\Delta\zeta + g(u^*)\zeta) \, dx \, dt = \int_\Omega \zeta(x,0) \, d\mu - \iint_{Q_T} \zeta \, d\lambda. \tag{2.47}$$

Since $g_k(u_k)$ converges to $g(u^*)$ uniformly in $Q_{\epsilon,T}$ for any $\epsilon > 0$, the measure λ is concentrated on $\overline{\Omega} \times \{0\}$. We denote by $\tilde{\lambda}$ its restriction to $\Omega \times \{0\}$, set

$$\mu^* = \mu - \tilde{\lambda},$$

and derive from (2.47),

$$\iint_{Q_T} (-u^*\partial_t\zeta - u^*\Delta\zeta + g(u^*)\zeta)\,dx\,dt = \int_\Omega \zeta(x,0)\,d\mu^*. \tag{2.48}$$

This implies $u^* = u_{\mu^*}$ and $Tr_\Omega(u^*) = \mu^*$, thus μ^* is a positive measure. Let v be a nonnegative subsolution of problem (2.2). By Proposition 2.8 there exists $\tilde{\mu} \in \mathfrak{M}^1_+(\Omega)$ such that $Tr_\Omega(v) = \tilde{\mu}$ and $\tilde{\mu} \le \mu$. Since $g_k(v) \le g(v)$, u is a subsolution for problem (2.43). By Theorem 2.11 $v \le u_k := u_{k,\mu}$. Thus $\lim_{k\to\infty} u_k = u^* \ge v$. □

Theorem 2.13 *The reduced measure μ^* is the largest good measure smaller than μ.*

Proof. Clearly μ^* is a good measure smaller than μ. Assume now that $\tilde{\mu}$ is a good measure smaller than μ. Then $u_{\tilde{\mu}}$ is a subsolution for problem (2.2). By (Theorem 2.11) u_{μ^*} is larger than $u_{\tilde{\mu}}$. Thus $Tr_\Omega(u_{\tilde{\mu}}) = \tilde{\mu} \le Tr_\Omega(u_{\mu^*}) = \mu^*$. □

The next technical result characterizes the good measures

Theorem 2.14 *Let $\mu \in \mathfrak{M}_+(\Omega)$. Then $\mu \in \mathcal{G}^\Omega(g)$ if and only if $g_k(u_k) \to g(u)$ in the weak sense of measures in $\mathfrak{M}^1(Q_T)$.*

Proof. Assume $g_k(u_k) \to g(u)$ in the weak sense of measures in $\mathfrak{M}^1(Q_T)$. Letting $k \to \infty$ in (2.33), we obtain (2.2) for any $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$. Thus $u^* = u_\mu$. Thus $\mu^* = \mu$ and μ is a good measure. Conversely, assume μ is a good measure. By Theorem 2.13, $\mu^* = \mu$. Thus $u_k \to u^* = u_\mu$ and $u_k \to u_\mu$ in $L^1(\Omega)$ and a.e. in Ω. Assume $\zeta \in C^{2,1}_{\ell,0}(\overline{Q}_T)$, $\zeta \ge 0$. We let $k \to \infty$ in (2.33) and derive

$$\lim_{k\to\infty} \iint_{Q_T} g_k(u_k)\zeta\,dx\,dt = \int_\Omega \zeta\,d\mu + \iint_{Q_T} (\partial\zeta + \Delta\zeta)\,u_\mu\,dx\,dt = \iint_{Q_T} g(u_\mu)\zeta\,dx\,dt, \tag{2.49}$$

by (2.2). Because $\{g_k(u_{k)}\}$ is uniformly bounded in $L^1_\rho(Q_T)$, the result follows by density. □

As in [4] an easy consequence of Theorem 2.13 is the following result which points out the fact that μ and μ^* differ only on a set with zero N-dimensional Hausdorff measure.

Corollary 2.15 *Let $\mu \in \mathfrak{M}^1_+(\Omega)$. There exists a Borel set $E \subset \Omega$, with Hausdorff measure $H^N(E) = 0$, such that $(\mu - \mu^*)(E^c) = 0$.*

Proof. Let $\mu = \mu_r + \mu_s$ be the Lebesgue decomposition of μ, μ_r (resp. μ_s) being the absolutely continuous (resp. singular) part relative to the Hausdorff measure H^N in $\mathbb{R}^N$. Both measures are positive. Since $\mu_r \in L^1_\rho(\Omega)$, it is a good measure. Then $\mu_r \le \mu^*$ by Theorem 2.13. Therefore

$$0 \le \mu - \mu^* \le \mu - \mu_r = \mu_s.$$

Since μ_s is singular relative to H^N, its support E satisfies $H^N(E) = 0$. This implies the claim. □

Corollary 2.16 *Let $\mu \in \mathfrak{M}^1_+(\Omega)$ such that $\mu(E) = 0$ for any Borel set $E \subset \Omega$ with $H^N(E) = 0$. Then μ is a good measure.*

Proof. Let $E \subset \Omega$ is a Borel set with $H^N(E) = 0$, then $\mu_r(E) = 0$. Since $\mu(E) = 0$, it implies $\mu_s(E) = 0$. Because the support of μ_s is a set with zero N-dimensional Hausdorff, $\mu = \mu_r = \mu^*$. □

Theorem 2.17 *Let $\mu_1, \mu_2 \in \mathfrak{M}^1_+(\Omega)$. If $\mu_1 \leq \mu_2$, then $\mu_1^* \leq \mu_2^*$. Furthermore*

$$\mu_2^* - \mu_1^* \leq \mu_2 - \mu_1. \tag{2.50}$$

Proof. For $k > 0$ let $u = u_{k,i}$ $(i = 1, 2)$ be the solution of

$$\left\{\begin{aligned} \partial_t u - \Delta u + g_k(u) &= 0 \quad \text{in } Q_T \\ u &= 0 \quad \text{in } \partial_\ell Q_T \\ u(.,0) &= \mu_i \quad \text{in } \Omega. \end{aligned}\right. \tag{2.51}$$

Since $\mu_1 \leq \mu_2$, $u_{k,1} \leq u_{k,2}$. By the convergence result of Theorem 2.12, the relaxed solutions u_i^* satisfies $u_1^* \leq u_2^*$. Since $\mu_i^* = Tr_\Omega(u_i^*)$, it follows $\mu_1^* \leq \mu_2^*$. We turn now to the proof of (2.50). If $\zeta \in C^{2,1}(\bar{Q}_t)$, $\zeta \geq 0$, which vanishes on $\partial_\ell Q_t$, we have from the weak formulation

$$\begin{aligned} \iint_{Q_t} (-(u_{k,2} - u_{k,1})(\partial_t\zeta + \Delta\zeta) + \zeta(g_k(u_2^*) - g_k(u_2^*)))\, dx\, dt \\ = \int_\Omega \zeta(x,0)d(\mu_2 - \mu_1) - \int_\Omega \zeta(x,t)(u_{k,2} - u_{k,1})dx \end{aligned}$$

We fix $\xi \in C_0^2(\bar{\Omega})$, $\xi \geq 0$ and choose for ζ the solution of

$$\left\{\begin{aligned} \partial_t\zeta + \Delta\zeta &= 0 \qquad \text{in } Q_t \\ \zeta &= 0 \qquad \text{on } \partial_\ell Q_t \\ \zeta(x,t) &= \xi \qquad \text{in } \Omega, \end{aligned}\right. .$$

Then, letting $k \to \infty$, we derive

$$\int_\Omega (u_2^* - u_1^*)(x,t)\xi dx \leq \int_\Omega \zeta(x,0)d(\mu_2 - \mu_1).$$

Finally, if $t \to 0$, using the trace property and the fact that $\zeta(x,0) \to \xi$ in $C_0(\bar{\Omega})$, we obtain

$$\int_\Omega \xi d(\mu_2^* - \mu_1^*) \leq \int_\Omega \xi d(\mu_2 - \mu_1).$$

This implies (2.50). □

Corollary 2.18 *If μ is a good measure, any positive measure ν smaller than μ is a good measure.*

Proof. Let $\nu \in \mathfrak{M}^1_+(\Omega)$, $\nu \leq \mu$. By (2.50)

$$0 \leq \nu - \nu^* \leq \mu - \mu^*.$$

Thus $\mu = \mu^* \Longrightarrow \nu = \nu^*$. $\square$

Corollary 2.19 *Let $\mu_1, \mu_2 \in \mathfrak{M}^1_+(\Omega)$. 1- If μ_1 and μ_2 are good measures, then so is $\inf\{\mu_1, \mu_2\}$ and $\sup\{\mu_1, \mu_2\}$.*

2- If $E \subset \Omega$ is a Borel set and $\mu \in \mathfrak{M}^1_+\Omega)$, $\mu^*|_E = [\mu|_E]^*$

3- Assume that μ_1 and μ_2 are mutually singular. Then $(\mu_1 + \mu_2)^* = \mu_1^* + \mu_2^*$.

Proof. 1- The fact that $\inf\{\mu_1, \mu_2\}$ is a good measure is clear from Corollary 2.18. Let $\nu = \sup\{\mu_1, \mu_2\}$. Then $\mu_1 \leq \nu^*$ and $\mu_2 \leq \nu^*$. Then $\nu = \sup\{\mu_1, \mu_2\} \leq \nu^*$.

2- We recall that $\mu|_E(A) = \mu(E \cap A)$, for any Borel subset A of Ω. We can also write $\mu|_E = \chi_E \mu$. Since $\mu \geq \mu^*$, $\chi_E \mu \geq \chi_E \mu^*$ and also $\mu^* \geq \chi_E \mu^*$. Thus $\chi_E \mu^*$ is a good measure and $[\chi_E \mu]^* \geq \chi_E \mu^*$ by Theorem 2.13. Conversely, $[\chi_E \mu]^* \leq \chi_E \mu$ implies that $\chi_E [\chi_E \mu]^* = [\chi_E \mu]^*$. But $\chi_E \mu \leq \mu$ implies $[\chi_E \mu]^* \leq \mu^*$ and therefore $[\chi_E \mu]^* = \chi_E [\chi_E \mu]^* \leq \chi_E \mu^*$.

3- If μ_1 and μ_2 are mutually singular, then so are μ_1^* and μ_2^*. Actually, $\mu_1 + \mu_2 = \sup\{\mu_1, \mu_2\}$ and $\mu_1^* + \mu_2^* = \sup\{\mu_1^*, \mu_2^*\}$. By assertion 1, $[\sup\{\mu_1^*, \mu_2^*\}]^* = \sup\{\mu_1^*, \mu_2^*\}$. Then $\mu_1^* + \mu_2^*$ is a good measure smaller than $\mu_1 + \mu_2$, thus $\mu_1^* + \mu_2^* \leq (\mu_1 + \mu_2)^*$. Conversely, there exist two disjoint Borel sets A and B such that $\mu_1 = \chi_A \mu_1$ and $\mu_2 = \chi_B \mu_2$ and $\mu_1 + \mu_2 = \chi_A \mu_1 + \chi_B \mu_2$. Thus $(\mu_1 + \mu_2)^* = (\chi_A \mu_1 + \chi_B \mu_2)^*$ and $\chi_A (\mu_1 + \mu_2)^* = (\chi_A \mu_1 + \chi_A \mu_2)^* = \chi_A \mu_1^* = \mu_1^*$. Similarly, $\chi_B (\mu_1 + \mu_2)^* = (\chi_B \mu_1 + \chi_B \mu_2)^* = \chi_B \mu_2^* = \mu_2^*$. Since

$$(\mu_1 + \mu_2)^* = \chi_{A\cup B} (\mu_1 + \mu_2)^* = \chi_A (\mu_1 + \mu_2)^* + \chi_B (\mu_1 + \mu_2)^*,$$

the result follows. $\square$

Theorem 2.20 *The set $\mathcal{G}^\Omega(g)$ is a convex lattice. Furthermore*

$$[\inf\{\mu, \nu\}]^* = \inf\{\mu^*, \nu^*\}, \tag{2.52}$$

and

$$[\sup\{\mu, \nu\}]^* = \sup\{\mu^*, \nu^*\}. \tag{2.53}$$

Proof. For the sake of completeness, we present the proofs of these assertions which actually the ones already given in [3]. Let $\mu_1, \mu_2 \in \mathcal{G}^\Omega(g)$ and $\nu = \sup\{\mu_1, \mu_2\}$. Since $\mu_i \leq \nu$, it follows from Theorem 2.17 that $\mu_i = \mu_i^* \leq \nu^*$. Thus $\sup\{\mu_1, \mu_2\} \leq \nu^*$ which reads $\nu \leq \nu^*$, and equality follows. Next, assume $\theta \in [0, 1]$. Then $\mu_\theta = \theta\mu_1 + (1-\theta)\mu_2 \leq \nu = \sup\{\mu_1, \mu_2\}$. Since $\nu \in \mathcal{G}^\Omega(g)$, and any measure dominated by a good measure is a good measure, $\mu_\theta \in \mathcal{G}^\Omega(g)$. It follows by Theorem 2.13 that $\mu_\theta = \mu_\theta^*$.

Next, by Corollary 2.19, $[\inf\{\mu^*, \nu^*\}]$ is a good measure. Since $[\inf\{\mu^*, \nu^*\}] \leq [\inf\{\mu, \nu\}]$, it follow by Theorem 2.13 that

$$\inf\{\mu^*, \nu^*\} \leq [\inf\{\mu, \nu\}]^*. \tag{2.54}$$

Conversely,

$$\inf\{\mu, \nu\} \leq \mu \Longrightarrow [\inf\{\mu, \nu\}]^* \leq \mu^*,$$

and similarly with ν. Thus $[\inf\{\mu,\nu\}]^* \le \inf\{\mu^*,\nu^*\}$.

For the last assertion, by Hahn's decomposition theorem there exist two disjoint Borel sets A and B such that $\Omega = A \cup B$ and $\sup\{\mu,\nu\} = \chi_A\mu + \chi_B\nu$. Actually, $\mu \ge \nu$ on A and $\nu \ge \mu$ on B. This implies also $\sup\{\mu^*,\nu^*\} = \chi_A\mu^* + \chi_B\nu^*$. Thus, by Corollary 2.19,

$$[\sup\{\mu,\nu\}]^* = (\chi_A\mu + \chi_B\nu)^* = \chi_A\mu^* + \chi_B\nu^* = \sup\{\mu^*,\nu^*\},$$

since $\sup\{\chi_A\mu^*, \chi_B\nu^*\} = \sup\{\mu^*,\nu^*\}$. $\square$

Theorem 2.21 *Let $\mu,\nu \in \mathfrak{M}^1_+$. Then*

$$|\mu^* - \nu^*| \le |\mu - \nu|\,. \tag{2.55}$$

Proof. We first assume $\mu \ge \nu$. By Theorem 2.17,

$$0 \le \mu^* - \nu^* \le \mu - \nu.$$

This implies (2.55). Next we write $\sup\{\mu,\nu\} = \nu + (\mu-\nu)_+$. Since $\nu \le \sup\{\mu,\nu\}$, $\nu^* \le [\sup\{\mu,\nu\}]^* = \sup\{\mu^*,\nu^*\}$ by Theorem 2.20. Thus

$$[\sup\{\mu,\nu\}]^* - \nu^* \le \sup\{\mu,\nu\} - \nu = (\mu-\nu)_+.$$

Thus implies $(\mu^* - \nu^*)_+ \le (\mu - \nu)_+$. Similarly $(\nu^* - \mu^*)_+ \le (\nu - \mu)_+$. $\square$

In order to characterize the universally good measures, we introduce a capacity naturally associated to the weak formulation of problem (2.2). This yields to a capacity type characterization of H^N. If $K \subset \Omega$ is compact, we denote

$$\begin{aligned} c_\Omega(K) = \inf\Bigg\{ \iint_{Q_T} |\partial_t\psi + \Delta\psi| \, dx\, dt :& \\ \psi \in C^{2,1}_{\ell,0}(\bar{Q}_T),\ \psi(x,0) \ge 1 \ \text{ in a neighborhood of } \ K &\Bigg\}. \end{aligned} \tag{2.56}$$

Theorem 2.22 *For every compact $K \subset \Omega$, we have*

$$H^N(K) = c_\Omega(K). \tag{2.57}$$

Proof. Let $K \subset \Omega$ be compact.

Step 1. We claim that for any $\epsilon > 0$, there exists $\psi_\epsilon = \psi \in C^{2,1}_{\ell,0}(\bar{Q}_T)$ such that $\psi \ge 0$ in Q_T, $\psi(x,0) \ge 1$ on K and

$$\iint_{Q_T} |\partial_t\psi + \Delta\psi| \, dx\, dt \le c_\Omega(K) + \epsilon. \tag{2.58}$$

Let $\xi \in C^{2,1}_{\ell,0}(\bar{Q}_T)$ such that $\xi(x,0) \ge 1$ on K and

$$\iint_{Q_T} |\partial_t\xi + \Delta\xi| \, dx\, dt \le c_\Omega(K) + \epsilon/2.$$

Let $\{\eta_j\}$ be a regularizing sequence depending only on the space variable and such that the support of η_j is contained in the ball B_{ϵ_j}, with $\epsilon_j \to 0$ as $j \to \infty$. If we extend ξ in $\mathbb{R}^N \times [0,T]$ as a $C^{2,1}$-function, we set

$$f_j(x,t) = \eta_j * |\partial_t \xi + \Delta \xi|(x,t) = \int_\Omega \eta_j(x-y)\,|\partial_t \xi + \Delta \xi|(y,t)dy.$$

If $j \to \infty$, $\{f_j\}$ converges to $|\partial_t \xi + \Delta \xi|$ uniformly in $\bar{Q}_T$. Let v_j be the solution of

$$\begin{cases} \partial_t v_j + \Delta v_j = -f_j \quad \text{in } Q_T \\ v_j = 0 \quad \text{in } \partial_\ell Q_T \\ v_j(.,T) = 0 \quad \text{in } \Omega. \end{cases}$$

Clearly $v_j \geq 0$ in Q_T. Let v be the solution of

$$\begin{cases} \partial_t v + \Delta v = -|\partial_t \xi + \Delta \xi| \quad \text{in } Q_T \\ v = 0 \quad \text{in } \partial_\ell Q_T \\ v(.,T) = 0 \quad \text{in } \Omega. \end{cases}$$

By the maximum principle $v \geq \max\{\xi, 0\}$, thus $v(x,0) \geq 1$ on K. Because $v_j(x,0) \to v(x,0)$ uniformly on $\bar{\Omega}$, for any $0 < \alpha < 1$, we can fix j_α such that $v_{j_\alpha}(x,0) \geq \alpha$ on K and $\|f_{j_\alpha}\|_{L^1(Q_T)} \leq \|\partial_t \xi + \Delta \xi\|_{L^1(Q_T)} + \epsilon/4$. Next $\psi_\alpha = \alpha^{-1} v_{j_\alpha}$. Then $\psi_\alpha \geq 0$ in Q_T, and $\psi_\alpha(x,0) \geq 1$ on K. Moreover

$$\begin{aligned} \iint_{Q_T} |\partial_t \psi_\alpha + \Delta \psi_\alpha|\,dx\,dt &= \alpha^{-1} \iint_{Q_T} |\partial_t v_{j_\alpha} + \Delta v_{j_\alpha}|\,dx\,dt \\ &\leq \alpha^{-1} \left(\iint_{Q_T} |\partial_t \xi + \Delta \xi|\,dx\,dt + \epsilon/4 \right) \\ &\leq \alpha^{-1} \left(c_\Omega(K) + 3\epsilon/4 \right). \end{aligned}$$

Next we fix

$$\alpha = \frac{c_\Omega(K) + 3\epsilon/4}{c_\Omega(K) + \epsilon}$$

and derive (2.58).

Step 2. There holds

$$H^N(K) \leq c_\Omega(K). \tag{2.59}$$

From (2.58),

$$\iint_{Q_T} (-\partial_t \psi - \Delta \psi)\,dx\,dt \leq \iint_{Q_T} |\partial_t \psi + \Delta \psi|\,dx\,dt \leq c_\Omega(K) + \epsilon.$$

But

$$\iint_{Q_T} (-\partial_t \psi - \Delta \psi)\,dx\,dt = \int_\Omega \psi(x,0)dx - \iint_{\partial_\ell Q_T} \frac{\partial \psi}{\partial n} dS\,dt \geq H^N(K)$$

since $\psi(x,T)=0$, $\psi(x,0)\geq 1$ on K, and the normal derivative of ψ on $\partial_\ell Q_T$ is nonpositive. This yields to (2.59) because ϵ is arbitrary.

Step 3. For any $\epsilon>0$ there exists $\psi\in C^{2,1}_{\ell,0}(\bar Q_T)$ such that $0\leq\psi\leq 1+\epsilon$ in Q_T, $\psi(x,0)\geq 1$ on K and

$$\iint_{Q_T}|\partial_t\psi+\Delta\psi|\,dx\,dt\leq H^N(K)+\epsilon. \tag{2.60}$$

For $\delta>0$ let $K_\delta=\{x\in\mathbb{R}^N:\operatorname{dist}(x,K)\leq\delta\}$. By the regularity of H^N, we can choose δ small enough such that

$$H^N(K_\delta\cap\Omega)\leq H^N(K)+\epsilon/5.$$

We fix $\xi\in C^{2,1}_{\ell,0}(\bar Q_T)$ such that $0\leq\xi\leq 1$ and

$$\xi(x,0)=\begin{cases}1 & \text{if } x\in K_{\delta/2}\\ 0 & \text{if } x\in\bar\Omega\setminus K_\delta.\end{cases}$$

Let $\sigma>0$ and

$$\rho_\sigma(x,t)=\left(1-\frac{t}{\sigma}\right)_+^2.$$

Since $|\xi_t+\Delta\xi|\,(x,t)=0$ a. e. on $\{(x,t):\xi(x,t)=0\}$ and $\{(x,t):\rho_\sigma(x,t)>0\}\subset\bar\Omega\times[0,\sigma]$, we can choose σ such that

$$\iint_{\partial_\ell Q_T\cap\{(x,t):\xi\leq\rho_\sigma\}}\frac{\partial\xi}{\partial n}dS\,dt+\iint_{\{(x,t):\xi\leq\rho_\sigma\}}|\xi_t+\Delta\xi|\,dx\,dt\leq\epsilon/5.$$

We set $u=\rho_\sigma-(\rho_\sigma-\xi)_+$. Because ρ_σ is independent of x, the argument developed by Brezis and Ponce [4] applies in the sense that $\Delta u(.,t)\in\mathfrak{M}(\Omega)$ and $\Delta u(.,t)=\Delta\xi(.,t)$ on $\{x:\xi(x,t)<\rho_\sigma(t)\}$ and more explicitely $\partial_t u+\Delta u=\partial_t\xi+\Delta\xi$ on $\{(x,t):\xi(x,t)<\rho_\sigma(t)\}$. In addition

$$\partial_t u=\partial_t\rho_\sigma-\operatorname{sign}_+(\rho_\sigma-\xi)(\partial_t\rho_\sigma-\partial_t\xi),$$

and $\partial_t u=\partial_t\rho_\sigma$ a.e. on $\{(x,t):\xi(x,t)\geq\rho_\sigma(x,t)\}$. Because ρ_σ is decreasing, we finally obtain

$$\partial_t u+\Delta u\leq 0 \text{ on } \{(x,t):\xi(x,t)\geq\rho_\sigma(x,t)\}.$$

We notice that $\partial_t u$ is bounded, and, following [4],

$$\begin{aligned}\|\partial_t u+\Delta u\|_{\mathfrak{M}}&=\left\|(\partial_t u+\Delta u)\chi_{\{\xi\geq\rho_\sigma\}}\right\|_{\mathfrak{M}}+\left\|(\partial_t u+\Delta u)\chi_{\{\xi<\rho_\sigma\}}\right\|_{\mathfrak{M}}\\&=\left\|(\partial_t u+\Delta u)\chi_{\{\xi\geq\rho_\sigma\}}\right\|_{\mathfrak{M}}+\iint_{\{\xi<\rho_\sigma\}}|\partial\xi+\Delta\xi|\,dx\,dt\\&=-\iint_{\{\xi\geq\rho_\sigma\}}d(\partial_t u+\Delta u)+\iint_{\{\xi<\rho_\sigma\}}|\partial\xi+\Delta\xi|\,dx\,dt\\&\leq-\iint_{Q_T}d(\partial_t u+\Delta u)+2\iint_{\{\xi<\rho_\sigma\}}|\partial\xi+\Delta\xi|\,dx\,dt\\&\leq-\iint_{Q_T}d(\partial_t u+\Delta u)+2\epsilon/5.\end{aligned}\tag{2.61}$$

Next, by definition,

$$
\begin{aligned}
-\iint_{Q_T} d(\partial_t u + \Delta u) &= -\iint_{Q_T} u(\partial_t 1 - \Delta 1)\,dx\,dt - \int_\Omega (u(x,T) - u(x,0))dx \\
&\qquad\qquad - \iint_{\partial_\ell Q_T} \frac{\partial u}{\partial n} dS\,dt \\
&= \int_\Omega u(x,0)dx - \iint_{\partial_\ell Q_T} \frac{\partial u}{\partial n} dS\,dt \\
&= \int_\Omega \xi(x,0)dx + \iint_{\partial_\ell Q_T \cap \{(x,t):\xi \le \rho_\sigma\}} \frac{\partial \xi}{\partial n} dS\,dt \\
&\le H^N(K_\delta)/ + \epsilon/5 \\
&\le H^N(K) + 2\epsilon/5.
\end{aligned}
\tag{2.62}
$$

We finally derive

$$
\|\partial_t u + \Delta u\|_{\mathfrak{M}} \le H^N(K) + 4\epsilon/5. \tag{2.63}
$$

Next, we smooth the measure $|\partial_t u + \Delta u|$ using a space convolution process with the same η_j, as in Step 1. One can construct a function $\psi \in C^{2,1}_{\ell,0}(\bar{Q}_T)$ such that $0 \le \psi \le 1 + \epsilon$ in $\bar{Q}_T$, $\psi(x,0) \ge 1$ on K and

$$
\iint_{Q_T} |\partial_t \psi + \Delta \psi|\,dx\,dt \le \|\partial_t u + \Delta u\|_{\mathfrak{M}} + \epsilon/5. \tag{2.64}
$$

Combining (2.63) and (2.64), one derive (2.60).

Step 4. There holds

$$
c_\Omega(K) \le H^N(K). \tag{2.65}
$$

Actually, (2.60) implies

$$
c_\Omega(K) \le H^N(K) + \epsilon.
$$

Letting $\epsilon \to 0$ yields to (2.65). □

Thanks to this result we are able to characterize the universally good measures.

Theorem 2.23 *Let $\mu \in \mathfrak{M}^1_+(\Omega)$. If $\mu \in \mathcal{G}^\Omega(g)$ for any function g satisfying (2.1), then $\mu \in L^1_\rho(\Omega)$.*

Proof. We follow essentially the proof of [4, Th 7].

Step 1. We claim that for every Borel set $\Sigma \subset \Omega$, such that $H^N(\Sigma) = 0$, there exists a continuous function g verifying (2.1) such that $\mu^* = 0$ for any $\mu \in \mathfrak{M}^1_+(\Omega)$ satisfying $\mu(\Sigma^c) = 0$.

Let $\{K_j\}_{j\in\mathbb{N}^*}$ be an increasing sequence of compact subsets of Σ such that $K = \cup_j K_j$ and $\mu(\Sigma \setminus K) = 0$. Since $H^N(K_j) = 0$ for any $j \ge 1$, it follows from Theorem 2.22 [Step 3], that there exists $\psi_j \in C^{2,1}_{\ell,0}(\bar{Q}_T)$ such that $0 \le \psi_j \le 2$ in Q_T, $\psi_j(x,0) \ge 1$ on K_j and

$$
\iint_{Q_T} |\partial_t \psi_j + \Delta \psi_j|\,dx\,dt \le 1/j.
$$

In particular,

$$|\partial_t\psi_j + \Delta\psi_j| \to 0 \quad \text{a.e. in } Q_T,$$

and, since ψ_j solves

$$\begin{cases} \partial_t\psi_j + \Delta\psi_j = \epsilon_j & \text{in } Q_T \\ \psi_j(x,T) = 0 & \text{in } \Omega \\ \psi_j(x,t) = 0 & \text{on } \partial_\ell Q_T \end{cases}$$

with $\epsilon_j \to 0$ in $L^1(Q_T)$ it follows $\psi_j \to 0$ in $L^1(Q_T)$ and a.e.. Furthermore there exists some $G \in L^1_\rho(Q_T)$ such that

$$\rho^{-1}\,|\partial_t\psi_j + \Delta\psi_j| \le G \quad \forall j \in \mathbb{N}^*.$$

By a theorem of De La Vallée-Poussin noticed in [5], there exists a convex function $h : (-\infty,\infty) \mapsto [0,\infty)$ such that $h(s) = 0$ for $s \le 0$, $h(s) > 0$ for $s > 0$,

$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty \quad \text{and } h(G) \in L^1_\rho(Q_T).$$

Let $g = h^*$ be the convex conjugate of h. We denote by $\mu^* = \mu^*(g)$ the reduced measured associated to g. Since $\mu^* \in \mathcal{G}^\Omega(g)$, we denote by u the solution of the corresponding initial value problem. Taking ψ_j as a test function in (2.2), we obtain

$$\iint_{Q_T} (-u(\partial_t\psi_j + \Delta\psi_j) + \psi_j g(u))\,dx\,dt = \int_\Omega \psi_j(x,0)d\mu^*. \tag{2.66}$$

We first assume that $\mu \in \mathfrak{M}^0(\Omega)$, thus we can take 1 as a test function (this is easily justified by approximations) and obtain

$$\iint_{Q_T} g(u)dx\,dt = \int_\Omega d\mu^*. \tag{2.67}$$

Therefore

$$\mu^*(K_j) \le \iint_{Q_T} (-u(\partial_t\psi_j + \Delta\psi_j) + \psi_j g(u))\,dx\,dt \tag{2.68}$$

and

$$\begin{aligned} |-u(\partial_t\psi_j + \Delta\psi_j) + \psi_j g(u)| &\le \frac{|\partial_t\psi_j + \Delta\psi_j|}{\rho} u\rho + \psi_j g(u) \\ &\le h\left(\rho^{-1}|\partial_t\psi_j + \Delta\psi_j|\right)\rho + g(u)\rho + \psi_j g(u) \\ &\le h(G)\rho + Cg(u) \end{aligned} \tag{2.69}$$

By Lebesgue's theorem, the right-hand side of (2.68) tends to 0 when $j \to \infty$. Thus $\mu^*(K_j) = 0$, for any $j \in \mathbb{N}^*$, and finally $\mu^*(\Sigma) = 0$.

Next we assume $\mu \in \mathfrak{M}^1(\Omega)$. Then there exists an increasing sequence of $\mu_n \in \mathfrak{M}^0(\Omega)$ with compact support in Ω such that $\mu_n \uparrow \mu$. Using what is proved above, $\mu_n^*(\Sigma) = 0$ and, by Theorem 2.17, $\mu^* \le \mu - \mu_n$, thus $\mu^*(\Sigma) \le (\mu - \mu_n)(\Sigma)$. Letting $n \to \infty$ implies $\mu^*(\Sigma) = 0$.

Step 2. If $\mu \in \mathfrak{M}^1_+(\Omega)$ is good, for any Borel set $\Sigma \subset \Omega$, with $H^{N-1}(\Sigma) = 0$, we denote $\nu = \mu|_\Sigma$. Then there exists g_ν such that $g_\nu^* = 0$. Since $\nu \le \mu$, $\nu \in \mathcal{G}^\Omega(g)$, thus $\nu = \nu^* = 0$ and finally, $\mu(\Sigma) = 0$. Thus $\mu \in L^1_\rho(\Omega)$. □

3 The Cauchy-Dirichlet problem

In this section Ω is again a smooth bounded domain in $\mathbb{R}^N$ and $\rho(x) = \mathrm{dist}\,(x, \partial\Omega)$. We denote by $\mathfrak{M}(\partial_\ell Q_T)$ the set of Radon measures in $\partial_\ell Q_T$ and by $\mathfrak{M}_+(\partial_\ell Q_T)$, the positive ones. The function g is supposed to satisfy (2.1). We consider the Cauchy-Dirichlet problem

$$\left\{\begin{aligned} \partial_t u - \Delta u + g(u) = 0 \quad & \text{in } Q_T := \Omega \times (0,T) \\ u = \mu \quad & \text{in } \partial_\ell Q_T := \partial\Omega \times (0,T) \\ u(.,0) = 0 \quad & \text{in } \Omega, \end{aligned}\right. \tag{3.1}$$

Definition 3.1 *Let $\mu \in \mathfrak{M}_+(\partial_\ell Q_T)$. A function $u \in L^1(Q_T)$ is a weak solution of (3.1) if $g(u) \in L^1_\rho(Q_T)$ and*

$$\iint_{Q_T} (-u\partial_t \zeta - u\Delta\zeta + \zeta\, g(u))\, dx = -\int_{\partial_\ell Q_T} \frac{\partial \zeta}{\partial \nu} d\mu, \tag{3.2}$$

for every $\zeta \in C_0^{2,1}(\bar{Q}_T)$.

Solutions of (3.1) are always unique; sufficient conditions for existence are developed in [8]. We define, similarly to the cases of the initial value problem, super and subsolutions of 3.1 . In which case, the equality sign in 3.2 is replaced by $\geq$ and $\leq$ respectively, the integrability conditions on u and $g(u)$ being preserved. As simple example for existence of a solution it is the case when g satisfies

$$\iint_{Q_T} g(\mathbb{P}^H[\mu](x,t))\rho(x) dx\, dt < \infty. \tag{3.3}$$

In this formula $\mathbb{P}^H[\mu]$ is the Poisson-heat potential of μ in Q_T, that is the solution of

$$\left\{\begin{aligned} \partial_t v - \Delta v = 0 \qquad & \text{in } Q_T \\ v = \mu \qquad & \text{in } \partial_\ell Q_T \\ v = 0 \qquad & \text{in } \Omega. \end{aligned}\right. \tag{3.4}$$

Definition 3.2 *A measure μ for which problem (3.1) can be solved is called a good measure relative to g for the Cauchy-Dirichlet problem. The set of good measures is denoted by $\mathcal{G}^{\partial_\ell Q_T}(g)$, and a universally good measure is a measure which belongs to $\mathcal{G}^{\partial_\ell Q_T}(g)$ for any g satisfying (2.1).*

The notion of lateral trace is defined in [9]. For $\beta > 0$, we denote

$$\Omega_\beta = \{x \in \Omega : \rho(x) < \beta\},\ \Omega'_\beta = \{x \in \Omega : \rho(x) > \beta\} \text{ and } \Sigma_\beta = \partial\Omega_\beta.$$

We shall also denote $\Sigma = \Sigma_0 = \partial\Omega$. There exists $\beta_0 > 0$ such that for any $\beta \in (0, \beta_0]$, the mapping $x \in \Omega_\beta \mapsto (\sigma(x), \rho(x)$, where $\sigma(x)$ is the unique point on $\partial\Omega$ which minimizes the distance from x to $\partial\Omega$, is a C^2 diffeomorphism from $\bar{\Omega}_\beta$ to $\Sigma \times [0, \beta_0]$. If $\phi \in L^1_{loc}(\partial_\ell Q_T)$, we denote $\phi^\beta(x,t) = \phi(\sigma(x), t)$, for any $x \in \Sigma_\beta$ and dS_β is the surface measure on Σ_β. for the sake of simplicity

Definition 3.3 *Let $u \in L^1_{loc}(Q_T)$. 1- We say that u admits the Radon measure $\mu \in \mathfrak{M}_+(\partial_\ell Q_T)$ as a lateral boundary trace if it exists*

$$\operatorname{ess\,lim}_{\beta\to 0} \int_0^{\mathrm{T}} \int_{\Sigma_\beta} \mathrm{u}\phi^\beta \,\mathrm{dS}_\beta \mathrm{dt} = \iint_{\partial_\ell \mathrm{Q_T}} \phi \,\mathrm{d}\mu \quad \forall \phi \in \mathrm{C}_0(\mathcal{R}). \tag{3.5}$$

We shall denote $\mu = Tr_{\partial_\ell Q_T}(u)$.
2- We say that u admits the outer regular Borel measure $\nu \approx (\Sigma, \mu)$ as a lateral boundary trace if it exists an open subset $\mathcal{R} \subset \partial_\ell Q_T$ and $\mu \in \mathfrak{M}_+(\mathcal{R})$ such that

$$\operatorname{ess\,lim}_{\beta\to 0} \int_0^{\mathrm{T}} \int_{\Sigma_\beta} \mathrm{u}\phi^\beta \,\mathrm{dS}_\beta \mathrm{dt} = \infty \quad \forall \phi \in \mathrm{C}_0(\partial_\ell \mathrm{Q_T}), \phi \geq 0, \phi > 0 \text{ somewhere in } \mathcal{S}, \tag{3.6}$$

with $\mathcal{S} = \partial_\ell \setminus \mathcal{R}$. We shall denote $\nu = Tr_{\partial_\ell Q_T}(u)$.

Propositions 2.5, 2.6, 2.7, 2.8 and Theorem 2.11 are still valid, if we replace the notion of initial trace by the notion of lateral boundary trace.The new version of Theorem 2.11 is the following.

Theorem 3.4 *Let u be a nonnegative subsolution of (3.1). Then the lateral boundary trace of u is a positive Radon measure $\tilde{\mu}$ such that $\tilde{\mu} \leq \mu$. Furthermore, if (3.1) admits a weak solution u_μ there holds $u \leq u_\mu$.*

We consider now a sequence of functions g_k satisfying (2.42). For any positive Radon measure μ on $\partial_\ell Q_T$, it is possible to solve, with $u = u_k$,

$$\left\{\begin{aligned} \partial_t u - \Delta u + g_k(u) = 0 &\quad \text{in } Q_T \\ u = \mu &\quad \text{in } \partial_\ell Q_T \\ u(.,0) = 0 &\quad \text{in } \Omega. \end{aligned}\right. \tag{3.7}$$

The following result is proved as Theorem 2.12

Theorem 3.5 *When $k \to \infty$, the sequence $\{u_k\}$ converges in $L^1(Q_T)$ to a some nonnegative function u^* such that $g(u^*) \in L^1_\rho(Q_T)$ and there exists a positive Radon measure μ^* smaller that μ with the property that*

$$\left\{\begin{aligned} \partial_t u^* - \Delta u^* + g(u^*) = 0 &\quad \text{in } Q_T \\ u^* = \mu^* &\quad \text{in } \partial_\ell Q_T \\ u^*(.,0) = 0 &\quad \text{in } \Omega. \end{aligned}\right. \tag{3.8}$$

Furthermore u^* is the largest subsolution of problem (3.1).

Mutadis mutandis, the reduced measure μ^* on the lateral boundary inherits the properties of the reduced measure at initial time and the assertions of Theorems 2.13, 2.14, Corollaries 2.15, 2.16, Theorem 2.17, Corollaries 2.18, 2.19 and Theorems 2.20 and 2.21, are valid in the framework of the lateral bondary reduced measure. The main novelty is the intruction of a new capacity on $\partial_\ell Q_T$. If $K \subset \partial_\ell Q_T$ is compact, we denote

$$c_{\partial_\ell Q_T}(K) = \inf\left\{\iint_{Q_T} |\partial_t\psi + \Delta\psi|\, dx\, dt \right.\\ \left. : \psi \in C^{2,1}_{\ell,0}(\bar{Q}_T),\ -\frac{\partial\psi}{\partial\nu} \geq 1 \text{ in some neighborhood of } K\right\}. \tag{3.9}$$

Theorem 3.6 *For every compact $K \subset \partial_\ell Q_T$, we have*

$$H^N(K) = c_{\partial_\ell Q_T}(K). \tag{3.10}$$

Proof. Let $K \subset \partial_\ell Q_T$ be compact.
Step 1. For any $\epsilon > 0$ there exists $\psi \in C^{2,1}_{\ell,0}(\bar{Q}_T)$, $\psi \geq 0$ such that $-\partial\psi(x,t)/\partial\nu \geq 1$ in some neighborhood of K.
Let $\xi \in C^{2,1}_{\ell,0}(\bar{Q}_T)$ such that $-\partial\psi(x,t)/\partial\nu \geq 1$ on K and

$$\iint_{Q_T} |\partial_t\xi + \Delta\xi|\, dx\, dt \leq c_{\partial_\ell Q_T}(K) + \epsilon/2.$$

We extend ξ as a $C^{2,1}(\mathbb{R}^N \times [0,T])$-function and define f_j, v_j and v in the same way as in the proof of Theorem 2.22, Step 1. Since $f_j \to \partial_t\xi + \Delta\xi$ uniformly in $\bar{Q}_T$,

$$\frac{\partial v_j}{\partial\nu} \to \frac{\partial v}{\partial\nu},$$

uniformly in $\bar{Q}_T$. Since v and ξ vanishes on $\partial_\ell Q_T$ and at $t = T$, $v \geq \xi$, thus

$$0 \leq \frac{\partial\xi}{\partial\nu} \leq -\frac{\partial v}{\partial\nu} \quad \text{on } \partial_\ell Q_T,$$

and $-\partial v/\partial\nu \geq 1$ in some neighborhood of K. For $\alpha \in (0,1)$ we fix j_0 such that $-\partial v_{j_0}/\partial\nu \geq \alpha$ on K and

$$\iint_{Q_T} |\partial_t v_{j_0} + \Delta v_{j_0}|\, dx\, dt \leq \iint_{Q_T} |\partial_t\xi + \Delta\xi|\, dx\, dt + \epsilon/4.$$

We set $\psi = \alpha^{-1}v_{j_0}$ and get

$$\iint_{Q_T} |\partial_t\xi + \Delta\xi|\, dx\, dt \leq \alpha^{-1}(c_{\partial_\ell Q_T}(K) + 3\epsilon/4).$$

We end the proof as in Theorem 2.22, Step 1.
Step 2. In this step we follow essentially the proof of [4, Lemma 8]. For any $\epsilon > 0$ there exists $\psi \in C^{2,1}_{\ell,0}(\bar{Q}_T)$, such that $0 \leq \psi \leq \epsilon$, $-\partial\psi(x,t)/\partial\nu \geq 1$ in some neighborhood of K and

$$\iint_{Q_T} |\partial_t\psi + \Delta\psi|\, dx\, dt \leq H^N(K) + \epsilon \quad \text{and} \quad \left|\frac{\psi}{\rho}\right| \leq 1 + \epsilon \text{ in } Q_T. \tag{3.11}$$

Let $\delta > 0$ and $\tilde{N}_\delta(K) = \{(x,t) : \mathrm{dist}\,((x,t),K)\}$, be such that

$$H^N(N_\delta(K) \cap \partial_\ell Q_T) \leq H^N(K) + \epsilon$$

We take $\xi \in C^{2,1}_{\ell,0}(\bar{Q}_T)$ such that $\xi > 0$ in Q_T, $\partial\xi/\partial\nu = -1$ on $N_{\delta/2}(K) \cap \partial_\ell Q_T$ and $\partial\xi/\partial\nu = 0$ on $\partial_\ell Q_T \setminus N_\delta$, $0 \leq -\partial\xi/\partial\nu \leq 1$ and $\xi/\rho \leq 1 + \epsilon$, we first take $a > 0$ small enough so that

$$\iint_{\partial_\ell Q_T \cap \{\xi < a\}} \frac{\partial\xi}{\partial\nu} dS\, dt + \iint_{Q_T \cap \{\xi < a\}} |\partial_t \xi + \Delta\xi|\, dx\, dt < \epsilon,$$

and set $u = a - (a - \zeta)_+$. Then, the same method as in Theorem 2.22-Step 3 yields to

$$\|\partial_t u + \Delta u\|_{\mathfrak{M}} \leq H^N(K) + 4\epsilon/5. \tag{3.12}$$

The conclusion of the proof is similar. □

By an easy adaptation of the proof of Theorem 2.23 we have the following characterization of the universally good measures.

Theorem 3.7 *Let $\mu \in \mathfrak{M}_+(\partial_\ell Q_T)$. If $\mu \in \mathcal{G}^{\partial_\ell Q_T}(g)$ for any function g satisfying (2.1), then $\mu \in L^1(\partial_\ell Q_T)$.*

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